Measurable Entanglement for Tripartite Quantum Pure States of Qubits

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We show that for tripartite quantum pure states of qubits, all the kinds of entanglement in terms of SLOCC classification are experimentally measurable by simple projective measurements, provided that four copies of the composite quantum system are available. In particular, the entanglement of reduced density matrices, even though they are mixed states, can be exactly determined in experiment. Concurrence of assistance is also shown to be measurable by introducing an interesting equations with explicit physical meanings.

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Introduction.- Quantum entanglement is not only one of the fundamental characteristics that distinguishes the quantum from the classical world, but also an important physical resource for quantum information processing. A lot of measures have been presented to quantify entanglement [1]. However, up till now, no directly measurable observable corresponds to entanglement of a given arbitrary quantum state, owing to the unphysical quantum operations in usual entanglement measure [2], for example, the complex conjugation of concurrence [3] and the partial transpose of negativity [4,5]. To evaluate the entanglement in experiment, a general approach is to reconstruct the density matrix by measuring a complete set of observables [6-8], which is only suitable for small quantum systems . Entanglement witnesses have been proven effect for the detection of entanglement [9], however they depend on the detected states, which implies that a priori knowledge on the states is required. Quite recently, some approaches have been reported for the determination of entanglement in experiment [2,10-12]. The most remarkable one is the new formulation of concurrence [13] in terms of copies of the state which led to the first direct experimental evaluation of entanglement [12]. Later, a measurable multipartite concurrence in terms of a single factorizable observable was presented [14]. The concurrence in terms of copies of states was generalized to mixed states [15], which in fact provides an observable lower bound of concurrence of mixed states and could be understood as a generalized entanglement witness. A natural problem is whether the tripartite entanglement is experimentally measurable.

Unlike bipartite entanglement which can be quantified by only a single quantity due to that any state can be prepared from a maximally entangled state by means of local operations and classical communication (LOCC), in general, tripartite entanglement can not be effectively quantified by a single scalar quantity because three qubits can be entangled in different ways [16]. It is obvious that different kinds of entanglement of a tripartite pure states can not be experimentally determined by the expectation of a single observable. In this Letter, we show that for an arbitrary tripartite quantum pure state of qubits it is possible to directly measure all the different kinds of entanglement based on four different projective measurements, provided that four copies of the tripartite quantum pure state are available. In particular, even the reduced density matrices are mixed, the exact entanglement instead of lower bound can be experimentally determined. In addition, an interesting equation with explicit physical meanings has been introduced by which we show that concurrence of assistance (COA)[17,18] is measurable.

Description of entanglement of tripartite quantum pure states of qubits.- A tripartite quantum pure states of qubits defined in the Hilbert space $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2\otimes\mathcal{H}_3$ can be written in standard basis by

$$\left|\psi\right\rangle_{ABC}=\sum_{i,j,k=0}^{1}a_{ijk}\left|i\right\rangle_A\left|j\right\rangle_B\left|k\right\rangle_C. \qquad (1)$$

It can be divided into six inequivalent classes under stochastic local operations and classical communication (SLOCC) [16], i.e. (i)unentangled states (tripartite separable states), if $\left|\psi\right\rangle_{ABC}=\left|\phi\right\rangle_A\otimes\left|\chi\right\rangle_B\otimes\left|\eta\right\rangle_C$; (ii)A-to-(BC) bipartite separable states, if $\left|\psi\right\rangle_{ABC}=\left|\phi\right\rangle_A\otimes\left|\varphi\right\rangle_{BC}$; (iii)B-to-(AC) bipartite separable states, if $\left|\psi\right\rangle_{ABC}=\left|\phi\right\rangle_B\otimes\left|\varphi\right\rangle_{AC}$; (iv)C-to-(AB) bipartite separable states, if $\left|\psi\right\rangle_{ABC}=\left|\phi\right\rangle_{AB}\otimes\left|\varphi\right\rangle_C$; (v) GHZ-type genuine tripartite entangled states with the standard form

$$\left|GHZ\right\rangle=\frac{1}{\sqrt{2}}\left(\left|000\right\rangle+\left|111\right\rangle\right), \qquad (2)$$

and (vi) W-type genuine tripartite entangled states with the standard form

$$\left|W\right\rangle=\frac{1}{\sqrt{3}}\left(\left|001\right\rangle+\left|010\right\rangle+\left|100\right\rangle\right). \qquad (3)$$

A direct and complete description of entanglement of tripartite quantum pure states of qubits is to define a four dimensional vector named entanglement vector which can be used to distinguish and quantify all the kinds of entanglement. For instance, define

$$\mathcal{E}(\left|\psi\right\rangle_{ABC})=\left[E_{ii},E_{iii},E_{iv},E_v\right], \qquad (4)$$

where $E_{ii} = C(|\psi\rangle_{A(BC)})$ denotes A-to-(BC) bipartite concurrence; E_{iii} and E_{iv} corresponds to B-to-(AC) and C-to-(AB) bipartite concurrence, respectively; E_v denotes the 3-tangle introduced in Ref. [19]. It is obvious that $\mathcal{E} = 0$ corresponds to class (i); $E_m = 0, m = ii, iii, iv$ corresponds to the mth class. $E_v \neq 0$ shows the existence of GHZ-type entanglement. Furthermore, let ρ_{AB} denote the reduced density matrix of two qubits, following the remarkable Coffman-Kundu-Wootters equation [19]

$$E_v + C^2(\rho_{AB}) + C^2(\rho_{AC}) = C^2(|\psi\rangle_{A(BC)}), \qquad (5)$$

and those corresponding to other foci, one can always determine the entanglement of reduced density matrices in terms of $\mathcal{E}$. Thus the existence of W-type entanglement can also be determined because the W-type relevant entanglement measure can always be given in terms of the entanglement of reduced density matrices [20]. In a word, so long as the entanglement vector $\mathcal{E}$ is given, all the kinds of entanglement can be determined. That is to say, if $\mathcal{E}$ is measurable, all the entanglement including those of the mixed reduced density matrices can be exactly determined in experiment. Ref. [14] has implied that the bipartite concurrence such as $C(|\psi\rangle_{A(BC)})$ is measurable by a simple projective measurement if two copies of the state are available, therefore all the remaining are to prove 3-tangle can also be measurable.

Measurable 3-tangle.-3-tangle can be defined by

$$\tau(|\psi\rangle_{ABC}) = 4\left|\det R\right|, \qquad (6)$$

where

$$R_{ij} = \langle\psi^*|_{ABC}\left(\sigma_y \otimes \sigma_y \otimes |i\rangle\langle j|\right)|\psi\rangle_{ABC}, \qquad (7)$$

with $i, j = 0, 1$. $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\{|i\rangle\}$ denotes the magic basis of $\mathcal{H}_3$. Consider the fourfold copy $\otimes_{k=1}^{4}|\psi\rangle_{ABC}$ of $|\psi\rangle_{ABC}$, one can define

$$P_-^{(i_m i_n)} = \frac{1}{\sqrt{2}}\left(|0\rangle_{i_m}|1\rangle_{i_n} - |1\rangle_{i_m}|0\rangle_{i_n}\right), \qquad (8)$$

denoting the projector onto the anti-symmetric subspace $\mathcal{H}_{i_m} \wedge \mathcal{H}_{i_n}$ of $\mathcal{H}_{i_m} \otimes \mathcal{H}_{i_n}$ where $i = A, B, C$ corresponds to the subsystems, and $m, n = 1, 2, 3, 4$ marks the different copies of $|\psi\rangle_{ABC}$. Thus a novel definition of 3-tangle can be derived through the expectation value of a self-adjoint operator $\mathcal{A}$ as

$$\tau(|\psi\rangle_{ABC}) = \sqrt{256\left(\otimes_{k=1}^{4}\langle\psi|_{ABC}\right)\mathcal{A}\left(\otimes_{k=1}^{4}|\psi\rangle_{ABC}\right)}, \qquad (9)$$

where $\mathcal{A}$ can be formally written by

$$\mathcal{A} = \left[\bigotimes_{\substack{k=A,B \\ j=1,3}} P_-^{(k_j k_{j+1})}\right] \otimes P_-^{(C_1 C_3)} \otimes P_-^{(C_2 C_4)}. \qquad (10)$$

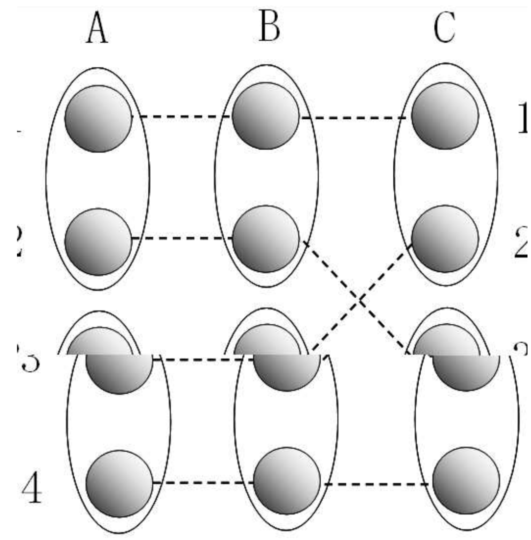

FIG. 1: The illustration of projective measurements for 3-tangle. The dotted line denotes the existence of quantum correlation. Every three balls connected by dotted lines denotes three particles in a copy of $|\psi\rangle_{ABC}$ with the number 1 to 4 marking the different copies. A, B and C on the top show that the three particles in a column correspond to the same subsystem. Every particle can be uniquely denoted by a letter and a number, for example A_1. Every loop represents a projective measurement P_- performed on the two particles inside. Even though a fourfold copy is necessary for 3-tangle, the projective measurements are only performed on a twofold copy.

$\mathcal{A}$ is obviously a single factorizable observable. Hence, $\tau(|\psi\rangle_{ABC})$ can be directly measured in experiment through projective measurements of the antisymmetric component of the twofold copy $|\psi\rangle_{ABC} \otimes |\psi\rangle_{ABC}$ among the four copies. An illustration of the projective measurements is depicted in Fig. 1. The analogous projective measurement has been demonstrated for twin photons in experiment [12].

Measurable concurrence of assistance.-Besides the entanglement classified under SLOCC mentioned above, there are another two important entanglement measures for tripartite quantum states, as far as we know. One is the global entanglement which is defined the same to tripartite concurrence in Ref. [14] and in fact turned out to be measurable in Ref. [14], the other is the concurrence of assistance which will be shown to be measurable next by introducing an interesting equation [21].

For $|\psi\rangle_{ABC}$, COA can be defined [17,18] by

$$C_a^{(AB)}(|\psi\rangle_{ABC}) = Tr\sqrt{\sqrt{\rho_{AB}}\tilde{\rho}_{AB}\sqrt{\rho_{AB}}}, \qquad (11)$$

with $\tilde{\rho}_{AB} = (\sigma_y \otimes \sigma_y)\rho_{AB}^*(\sigma_y \otimes \sigma_y)$. $C_a^{(AB)}(|\psi\rangle_{ABC})$ maximizes the average concurrence shared by A and B with

the help of C. Let $C(\rho_{AB})$ denote the concurrence of the reduced density matrix ρ_{AB}, then what is the difference between $C_a^{(AB)}$ and $C(\rho_{AB})$? Let λ_1 and λ_2 be the square roots of the two eigenvalues of $\rho_{AB}\tilde{\rho}_{AB}$, then $C_a^{(AB)}$ can be rewritten by $\lambda_1+\lambda_2$ and $C(\rho_{AB})$ can be given by $|\lambda_1-\lambda_2|$. Thus the difference between them can be directly written by

$$\left[C_a^{(AB)}\right]^2 - C^2(\rho_{AB}) = 4\lambda_1\lambda_2 = \tau\left(|\psi\rangle_{ABC}\right). \qquad (12)$$

Consider the concurrence shared by two different parties among A, B and C, there exist another two analogous equations to eq. (12). Since both $C(\rho_{AB})$ and $\tau\left(|\psi\rangle_{ABC}\right)$ can be experimentally determined, COA is also measurable.

In fact, besides the main result that eq. (12) shows the measurable COA, eq. (12) also has explicit physical meanings. As we know, $C^2(\rho_{AB})$ denotes the entanglement of Parties A and B, and $\left[C_a^{(AB)}\right]^2$ is the maximal average entanglement shared by A and B with the help of C taken into account. Eq. (12) implies i) COA includes two contributions: concurrence of the two considered qubits and three-way entanglement; ii) The role of C is to convert the three-way entanglement shared by three parties into bipartite entanglement shared by two parties, thus entanglement shared by two parties is increased. Two most obvious examples are GHZ state and W state. The entanglement of reduced density matrix of GHZ state is zero, hence the COA of GHZ state all comes from the three-way entanglement and equals to 1 (the value of 3-tangle). On the contrary, the W state has no three-way entanglement (only two-way entanglement) [22], hence its COA is only equal to the concurrence ($\frac{4}{9}$) of the two parties. That is to say, for W state, party C can not provide any help to increase the entanglement between A and B.

An alternative description of entanglement of tripartite pure states.-One can find any four quantities will be valid for the entanglement vector if the four quantities can effectively distinguish and quantify all the kinds of entanglement of tripartite pure states. From the previous choice of the entanglement vector, it is not difficult to see that the entanglement vector must be completely determined in order to evaluate the entanglement of a single two-qubit reduced density matrix or a single COA. Hence, a more convenient description is expected. From eq. (11) and the expression of $C(\rho_{AB})$, one has

$$\left[C_a^{(AB)}\right]^2 = Tr\left(\rho_{AB}\tilde{\rho}_{AB}\right) + \frac{1}{2}\tau\left(|\psi\rangle_{ABC}\right), \qquad (13)$$

and

$$C^2(\rho_{AB}) = Tr\left(\rho_{AB}\tilde{\rho}_{AB}\right) - \frac{1}{2}\tau\left(|\psi\rangle_{ABC}\right). \qquad (14)$$

In particular, in terms of the twofold copy of $|\psi\rangle_{ABC}$ (or ρ_{AB}), one can get

$$Tr\left(\rho_{AB}\tilde{\rho}_{AB}\right) = \sqrt{Tr[(\rho_{AB}\otimes\rho_{AB})\mathcal{B}]}, \qquad (14)$$

where $\mathcal{B} = 4P_-^{A_1A_2}\otimes P_-^{B_1B_2}$. $Tr\left(\rho_{AB}\tilde{\rho}_{AB}\right)$ has been written in the form of the expectation value of the self-adjoint operator $\mathcal{B}$, hence it is measurable. Consider the other two pairs of equations for ρ_{AC} and ρ_{BC} and the CKW equations, one can determine all the entanglement, provided that two copies of the reduced density matrix of two qubits are available. Thus a new entanglement vector can be constructed by means of replacing E_{ii}, E_{iii}, E_{iv} by three $Tr\left(\rho_x\tilde{\rho}_x\right)$ with x denoting two qubits. With the new entanglement vector, it not necessary to know all the elements of the vector in order to determine a given entanglement except the global entanglement.

We have considered that the measured quantum states are pure. However, the imperfect preparation procedure may produce mixed states. In practical experiment, analogous to Ref. [2], one can discuss the deviation of measured entanglement by considering the potential errors introduced by impure states and correct the measurement values. Such an comparison procedure is quite simple and omitted here.

Summary.- We have shown that 3-tangle can be experimentally determined by a single factorizable observable, provided that four copies of the state can be provided, by which all the entanglement in terms of SLOCC classification can be determined with the help of the measurable bipartite concurrence or $Tr\rho_x\tilde{\rho}_x$. COA has also been shown to be measurable by an interesting equation with explicit physical meanings. We would like to emphasize that although reduced density matrices of two qubits are mixed states, the exact concurrence instead of the lower bound can be determined. Even though four copies of the state are required, all the projective measurements are only restrictive on the twofold copies, as has been demonstrated recently in experiment. Furthermore, because a state has to be prepared repeatedly in order to obtain reliable measurement statistics in any experiment [2], a fourfold copy of a state should be feasible in current experiment, which implies the observation of all the entanglement of tripartite pure states of qubits may be feasible.

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$$\overline{E}\left(|\psi\rangle_{ABC}\right) = \frac{1}{3}\left(C^2(\rho_{AB}) + C^2(\rho_{AC}) + C^2(\rho_{BC})\right),$$

and the least entanglement of the reduced density matrices defined by

$$E_{min}\left(|\psi\rangle_{ABC}\right) = \min\left(C^2(\rho_{AB}), C^2(\rho_{AC}), C^2(\rho_{BC})\right).$$

Another alternative measure we propose can be given by

$$E\left(|\psi\rangle_{ABC}\right) = \left(C^2(\rho_{AB})C^2(\rho_{AC})C^2(\rho_{BC})\right)^{1/3}.$$

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